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With which is incorporated The Institution of Aeronautical Engineers

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BY

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THE ROYAL AERONAUTICAL SOCIETY

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with which is incorporated the Institution of Aeronautical Engineers

VOLUME XLVI

MONTHLY NOTICES.

JANUARY. 1942.

New Year.

The President and Council wish all members, here and overseas, a safe New Year, with the hope that it will see the dawn of peace lighting up the sky.

New Year's Honours.

Honour of Knighthood.

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C.B.

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A. P. Rowe, Fellow.

A.F.C.

Acting Squadron Leader A. M. Jardine, Student.

Mentioned in Despatches.

Air Commodore H. V. Rowley, Associate Fellow.
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Annual Subscriptions.

Members are reminded that their subscriptions are due on January 1st, 1942. The following are the current rates:—

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Journal.

The particular attention of members is drawn to the fact that the Journal is not sent to members whose current subscription has not been paid by March 31st. On account of the restriction of paper it will not be possible to print any extra supply of JOURNALS, and members whose subscriptions are in arrears cannot be guaranteed the back numbers of JOURNALS, as the printing order is now being kept down to its minimum.

Journal Binding.

The following is a list of revised prices for binding the Society's JOURNALS:—

1941 Volume, including Library Catalogue	6s. 6d.
1941 Volume, excluding Library Catalogue	5s. 6d.
Previous Volumes	6s. 6d.
Binding Library Catalogue separately from the JOURNAL ...	5s. 0d.

Advertisements.

The fact that goods made of raw materials in short supply, owing to war conditions, are advertised in the JOURNAL should not be taken as an indication that they are necessarily available for export.

Election of Members.

Fellows.—Donald Clifford Tyndall Bennett (from Associate Fellow), James Harold Doolittle (from Associate Fellow).

Associate Fellows.—Winston Foster, Kenneth Cecil Pratt (from Student), Pierre Zephirin Roux, George Ernest Watt, Donald Crawford Wood (from Student), Ronald James Brabant Woodhams.

Associates.—Percy de Leuville Dyson-Skinner, Charles Henry Ford, Thomas Eric Greenwood, Albert Edward O'Nions, Desmond John Tynan-Bird, Thomas Joseph Walsh.

Graduates.—John Beale, Jack Grant, John Kitchen Moakes (from Student), Walter Vacey.

Students.—John Alistair Peter Anson, Peter Frederick Ashwood, Raymond Rupert Caveney, Stuart James Cox, Richard Evans-Thomas, Thomas Vincent Foster, Sydney James Garvey, Peter William Gerrett Hargreaves, Eric Douglas Margetts, Ernest Frederick Needham, Anthony Noel Watson, Norman Arthur White.

Companions.—Philip James Futer, John Talbot-Rowe.

Additions to the Library.

Pamphlets in italics, with location reference following in brackets.
Books marked * may not be taken out on loan.

*BB.c.83.—Airscrew Performance Calculations. (De Havilland Aircraft Co., Report No. R.83.) By K. B. Gillmore, A. V. Cleaver and J. Mullin. 1941.

*E.c.23.—A Treatise on Photo-Elasticity. By E. G. Coker and L. N. G. Filon. Cambridge University Press. 1931. 52/6.

G.e.E.52.—*The Influence of Stress on the Corrosion Pitting of Steel in Distilled Water. By D. J. McAdam and G. W. Geil. Preprint of paper read at the 23rd Annual Meeting of the American Society for Metals, Philadelphia, October,* 1941. (PG. 6/17.)

J.b.13.—*India Meteorological Department: Scientific Notes, Vol. VIII*:—

No. 94. Photographic Studies of some Cloud Forms and their changes with time. By M. W. Chiplonkar. 1941. 2/3.

No. 96. Frequency of Thundery Conditions at Bombay, compared with those at some other Stations in India. By M. W. Chiplonkar. 1941. 9d.

**R.f.71.—*Photograph of Propaganda Pamphlet, scattered over Rome by Lauro de Bosis in October, 1931; together with typewritten lecture "Lauro de Bosis," by Paolo Vivante.* 1941.

UL.6.—*Aeronautical Research Institute, Tokyo. Reports*:—

No. 211. On the Theory of Turbulent Boundary Layer on a Flat Plate. By T. Sasaki.

No. 212. The Coupling of the Propeller Blade Bending Vibration with the Crankshaft Twisting Vibration of a V-Type Engine. By K. Tanaka and T. Ohino.

(Both in Japanese with brief summary in English.)

*V.5/45.—Institution of Mechanical Engineers. Proceedings, Vol. 145. (January-June, 1941.)

*X.c.E.3.—*Aeronautika Terminaro. (Glossary of Aeronautical Terms in Esperanto.) By E. D. Durrant. Internacia Esperanto-Ligo.* 1941. 10d.

*X.c.P.1.—*Anglo-Polish Translation of Workshop Terms. By W. Bastyr and E. Paszkowski. Association of Polish Engineers in Great Britain.* 1941. 3/-.

Y.24.—*Publications of the Tin Research Institute*:—

No. 106. The Longitudinal Ridged Structure in the Tin Coating of Tinplate. By B. Chalmers and W. E. Hoare.

Acknowledgments.

The President and Council acknowledge with grateful thanks the gift of JOURNALS of the Society from Group Captain Pankhurst and from the Honorary Secretary of the Coventry Branch of the Society.

J. LAURENCE PRITCHARD, *Secretary & Editor.*

THE CARTOGRAPHIC SOLUTION OF GREAT CIRCLE PROBLEMS.

By A. J. Dilloway, F.R.G.S.

1. Introduction.

During the last twenty years a revival of interest in the subject of map projections has taken the form of a renewed attack upon certain basic limitations which face the cartographer. A necessary stimulus has been provided by the growth of a technique of air navigation, which in many ways has departed from the traditional practice of the marine navigator. From this and other directions emphasis has been laid on the need for precise cartographic representation of distance, direction and position on the earth. But although certain problems may be treated by devising new projections to meet special needs, it is obvious that there is a limit to any *purely* cartographic approach.

The requirements of the new navigation are, briefly, that its methods should be rapid, convenient to use, and of an accuracy consistent with the limitations imposed. The traditional solution by means of spherical trigonometry is thus ruled out on at least two of these counts. In its place, there are now available a number of graphical, mechanical and simplified tabular methods, many of which have no cartographic basis. Since a map projection is an obvious medium for the measurement of spherical relations, the needs of long-distance air navigation have encouraged the adaptation of certain projections in the form of special instruments and devices. It should not be assumed, however, that the need for special methods is confined to the field of navigation. In general, it is obvious that, as a result of development in long-distance travel and communication, our geographical ideas will tend increasingly to become oriented along great circles of the earth, rather than along the " straight lines " of former days. Rapid and convenient solution of problems in distance, bearing and convergence is therefore becoming more important in many branches of geographical research. An interesting paper by Stevens[1] provides a good example of this tendency in the field of structural geology.

For present purposes, four possible methods of attacking spherical problems may be distinguished:—

1. The normal mathematical solution of the spherical triangle.
2. Graphical and mechanical methods and devices by which the mathematical solution may be simplified.
3. Solution by measurement between plotted positions on normal map projections.
4. Graphical and mechanical adaptations of certain map projections.

The present paper deals only with the third and fourth of these groups, which correspond respectively to what may be called a " pure " and an " applied " use of projections in measuring spherical relations. Although a good deal of work has been done on various aspects of such problems, the subject as a whole has received no systematic treatment. The objects of the present paper are, therefore, to attempt a comprehensive survey of various methods at present available, and to consider in relation to them certain new lines of approach which further study has suggested. Because the search is largely for simplicity and convenience of measurement, this discussion must be essentially a graphical one.

[1] Stevens, A. " A Preliminary Study of the World Geometry of Structure Lines." *Trans. Geol. Soc. Glasgow*, **17**, 440-463.

As would be expected, the best and most typical examples of every kind of practical problem are to be found in the work of the long-distance air navigator. For this reason the subject as a whole is treated from the navigational viewpoint. At the same time, a considerable range of methods is now available, and, at a risk of apparent overlapping, these are discussed in sufficient detail to allow a choice to be made for any particular purpose. The paper consists, therefore, of an introductory section, followed by a detailed study of various useful projections, and concluding with a summary of the uses to which certain instruments derived from these projections may be put.

* * * * *

All long-distance navigation requires :—

1. A preliminary determination of distance, course and great circle track.
2. The maintenance of the plane or ship on the track so determined.

Under the first heading we are concerned with the spherical relations of two points the positions of which are known. The second heading (so far as it concerns us) refers to the laying down of various forms of loci of position. Here we are dealing with elements required when the co-ordinates of one point are given and it is required to determine those of a second point known only approximately. In the first case the data are taken solely from maps, while in the second they are deduced both from terrestrial observations of bearing and from celestial observations of both bearing and altitude. A classification of the special problems of the navigator in a form suitable for general discussion may therefore be attempted as follows :—

A. Given the geographical co-ordinates of two points :—
   1. To determine the great circle (and loxodromic) distance between them.
   2. To determine the initial bearing of one from the other.
   3. To determine the convergence between their two meridians.
   4. To determine the elements of a great circle passing through them, *i.e.*, to determine the co-ordinates of selected points along its path, and the angles included at these points between the local meridian and a tangent to the great circle.

B. Given the geographical co-ordinates of one point :—
   1. To determine the locus of a second point when the distance between the two (*i.e.*, zenith distance) and a bearing of the known position, measured at an unfixed position, are given.
   2. To determine the locus of a second point when a bearing of an unfixed position, measured at the known position, is given.
   3. To determine the locus of a second point when a bearing of the known position, measured at an unfixed position, is given.

It should be noted that the measurement of areas is not considered, since this matter is completely covered by a correct use of appropriate graticules.

The projections to be discussed may evidently be arranged in relation to problems to be solved. They will thus be divided into two main groups, according as they provide solutions of normal spherical problems (Group A) or of a second group of position-line problems. By the use of a well-known convention these two ways of defining relations between points on the earth—the one by means of spherical co-ordinates and the other by the use of loci—are to some extent reduced to a common form. So much attention will be given to this idea that it may be stated fully at the outset. Thus, if we consider the distance and azimuth of a point A on the earth from another point B (Fig. 1*a*) the great circle distance between them may be expressed as " co-latitude " and the azimuth of A from B as " longitude " on the system of co-ordinates centred on pole B. In general, it is evident that a point, situated at any given latitude and longitude on the sphere, may become the " pole " of

an oblique system of great and small circles, the intersections of which will have co-ordinate values on the geographical network. If any projection of the geographical system be drawn, these intersections may be plotted upon it, forming an oblique case of the projection. When this oblique case is of cartographic value it may of course be used independently. The oblique Mollweide[2] is a case in point.

It is also possible to make use of the principle without actually considering two sets of co-ordinates. Suppose the distance AB and the azimuth of B from A to be required (Fig. 1*b*). If the extremities of the great circle AB be imagined to rotate about an axis XY, perpendicular to the plane of the meridian of A, the point A may be moved through its co-latitude to occupy the position of pole P. The point B will then occupy a position Q and the distance AB will equal the co-latitude PQ. The azimuth of B from A can be read off the graduation as difference of longitude APQ. These ideas have found cartographic expression in various familiar devices, ranging from the Baker Navigation Machine to the Reeves Azimuth Diagram and its orthographic equivalent. In general, it may be said that, just as the spherical triangle is the basis of the mathematical study of distance, direction and position on the earth, so the elementary ideas outlined above constitute the principles on which a cartographic approach to the subject can be based.

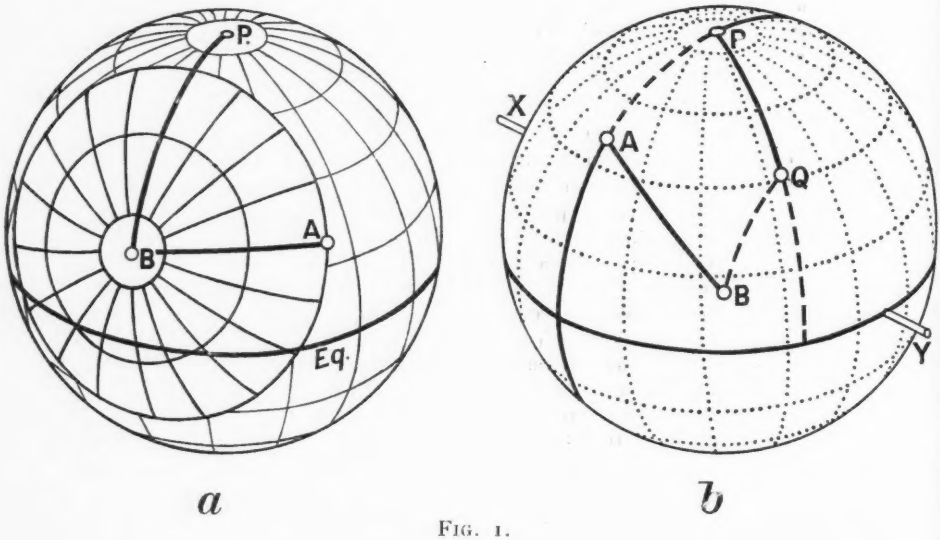

FIG. 1.

Projections may also be classified in accordance with the nature of the representation they provide. On the one hand types such as the conical, polyconic, minimum-error and other miscellaneous projections give a fair approximation to true spherical conditions over a limited area. On the other hand the Mercator, the zenithals and various special cases provide an exact representation of certain elements at the expense of loss of accuracy or inconvenience of measurement in other directions. This second group includes both projections which possess certain special properties in addition to their normal uses, and others specially designed for the solution of navigational and other problems and of little use for any other purpose.

[2] Close, Col. Sir C. "An Oblique Mollweide Projection of the Sphere." *Geog. Jour.* 73 (1929.)

It is evident that, so far as special adaptation is concerned, minimum-error and other approximate cases will be of little use, and may be neglected. Their contribution is rather to be assessed on the basis of the standard of accuracy required for a particular purpose, and of the extent of the area over which normal measurements may be expected to conform to such a standard. The use of projections of this kind therefore depends on principles already well established. The position regarding the adaptation of special properties is quite another matter. In this case, certain devices and methods, developed in response to the needs of the air navigator, provide isolated examples of a use of general principles, which, to a great extent, enable the limitations of particular projections to be overcome.

The choice of suitable graticules is largely concerned with the distribution of what may be called " lines of strength." Such lines are not necessarily great circles, nor are they always lines of true scale. On the contrary, orderly change of scale is the more important quality, for it enables additional properties to be introduced. By far the most useful framework of this kind is provided by the radial and circumferential scales of zenithal projections. It is evident that the frame of true scale representation provided by standard parallels and isoperimetric curves of certain conicals, or by the standard meridians of such projections as that of the International Map, will be of little use for accurate work, since the effect is merely to minimise the total errors of the map, rather than to preserve true conditions of distance and direction in respect of a given point. It may be laid down as a principle that graticules suitable for accurate determinations must provide either true representation of certain properties in respect of one or more points, or must preserve a given relation uniformly over the area covered. On the basis of the above remarks, the projections available for spherical measurements may be classifed as follows:—

1. Projections providing accurate representation of one or more elements of the sphere:—

   a. In respect of any single chosen point.
   All projections of the zenithal class.
   Craig's Retro-Azimuthal Projection.
   Retro-Azimuthal Equidistant Projection.

   b. In respect of any two chosen points.
   Two-Point Gnomonic Projection.
   Two-Point Equidistant Projection.
   Two-Point Retro-Azimuthal Projections.

   c. In respect of any point situated on the central meridian.
   Littrow Projection.
   Maurer's Orthomorphic Retro-Azimuthal Projection.

   d. In respect of the whole area of the map.
   Gnomonic Projection.
   Mercator's Projection (normal, oblique and transverse).

2. Projections providing an approximate and variable representation of distance and direction over the whole area of the map.
   Lambert's Conical Orthomorphic Projection.
   Albers' Conical Equal Area Projection.

Generally speaking, projections belonging to groups *b* and *c*, or to the second of the types distinguished, will be used in the normal way, and determinations will be made by direct measurement between points defined by their co-ordinates. The solution obtainable will, therefore, be either approximate but universally applicable, as in the case of the latter, or accurate, but applicable only to a restricted number of points, as in the former (*b* and *c*). On the whole, graticules like those of groups *a* and *d* lend themselves more readily to adaptation. In this way, the specialised conditions under which accuracy is obtainable may sometimes

be evaded, thus enabling true measurements to be made in respect of *any* two chosen points.

The graphical and mechanical adaptations of map projections so far devised are based on the application of four principles:—

a. Rotation of geographical co-ordinates by means of superimposed projections.

b. Graphical transformation of such co-ordinates.

c. Use of curves of equal linear or angular scale.

d. Use of loci of intersections of meridians and parallels.

The first and second of these methods are implicit in the conventions described above. For different purposes the first principle has been applied to both the Mercator and stereographic projections, while the application of the second through the medium of the stereographic is believed to be new. In addition to its use by Penfield, the third principle has been applied to the gnomonic projection. Its relation to the idea of rotation is at once apparent. So far as the present writer is aware, the use of loci in dealing with the intersections of a graticule is described for the first time.

2. Detailed Study of Projections.

a. Solution of Problems Involving Two Fixed Positions.

These problems are concerned with the determination of distance, azimuth and difference of azimuth between two points, and of the elements of the great circle joining them. The most useful projections for this purpose are those belonging to groups *a* and *d* in the above classification. Of these the stereographic, the gnomonic and the Mercator are by far the most important. For present purposes, the stereographic merits greater attention than it normally receives, and it is hoped to show that this much-neglected projection is by far the most important of the zenithals, and, everything considered, perhaps of all projections. Other graticules requiring more than a brief mention are those of Craig's Retro-Azimuthal and Lambert's Conical Orthomorphic projections. Since the stereographic furnishes the only graticule to which each of the four principles of adaptation enumerated above may be applied, it might be helpful to consider it first, and to deal with others in relation to it.

Stereographic Projection.

The properties which contribute to the importance of this projection for present purposes are:—

1. Great circles passing through a chosen point (the centre of the map) are preserved as straight lines, and include true azimuths with the central meridian.
2. Variation of scale along such great circles is uniform in any direction from the centre.
3. This regular variation of scale from a centre is uniform for all latitudes provided the scale of construction does not vary.
4. All other great and small circles on the sphere are preserved as circles on the map.
5. Meridians and parallels intersect at right angles, and make true angles with great circles passing through the centre.

Apart from the possibility of adaptation, this projection is evidently of value for two purposes. Within certain limits it may provide an approximate basis for measurement between any two points, or it may be used to give an accurate representation of distance, bearing and reverse bearing in respect of any one chosen point. Within a radius of some 500 miles from the centre, distances and bearings may be measured directly, and the special properties provide against

serious distortion. The term "stereodrome" has been applied by Prof. Immler[3] to any straight line drawn on a graticule of this kind. For measurements up to 1,000 miles the greatest error involved in considering such a line as a great circle is given by him as less than one per cent. Over the area concerned meridians may also be considered as straight lines inclined to each other at the true convergence angle, and true bearings may therefore be measured directly. The following method is recommended by Immler for the measurement of distance. A perpendicular may be dropped from the centre of projection on to the line to be measured. This line may then be turned about the centre until it occupies the position of a meridian. Great circle distance can then be measured as difference of latitude.

The idea of rotation is very obviously related to a simple and very useful adaptation of the stereographic long since proposed by Penfield.[4] If a suitable graticule be prepared for a given area, a corresponding grid of circular arcs (representing great circles and great circle distances) may be plotted as a transparency and superimposed. If now any two points (A and B, Fig. 2*a*) be plotted on the projection, this transparency may be turned about the respective centres until the two points lie on one of the great circles (ab in the figure). The intercept AB read off the orthogonal system of small circles will give the distance between the points. This method is free from error and, so far as actual measurement is concerned, extreme accuracy is possible. It has occurred to the author that the measurement of bearing, which in this case is less satisfactory than that of distance, would be much improved by making use of a circular protractor carrying a rotatable arm. If the extremities of this arm were also graduated it could be used to define the tangent, and therefore the angle required. Penfield's idea was intended for navigational as well as geographical purposes, but the writer is not aware that it has been put to any practical use.

The transparent system of great and small circles used in this method may be considered as the central portion of an equatorial case of the projection turned through 90°. The whole idea is thus seen to resemble a large-scale use of the principle of rotation, already applied to small scale stereographic projections in the form of the Reeves Azimuth Diagram. This latter device is intended primarily for the determination of the reverse azimuth required in setting an aerial for long-distance reception, but it may also be used for the small-scale solution of all problems of the normal type. Once again the underlying principle is simple and well-known. If two equatorial hemispheres of the stereographic be superimposed, their rotation about a common centre will enable the relations between oblique co-ordinates on the earth to be studied. Since azimuths from the centre are correctly given, the upper set of co-ordinates may be dispensed with, and the points themselves turned through any angle to any required position. Thus, if the relations between two points A and B (Fig. 2*b*) are required, A may be plotted at the appropriate latitude on the meridian of the bounding circle and B placed at the given latitude and given difference of longitude from A. The point B may now be turned about the centre C until it occupies the position B', the angle BCB' being equal to the co-latitude of A. The pole P will then represent the position of A, and the great circle distance AB will be read off the graduation as the difference of latitude PB'. Similarly, the azimuth of B from A becomes the difference of longitude between B' and the bounding circle. The co-ordinates of selected points $1'2'$ on the great circle AB may be determined by turning them back through the angular amount of the co-latitude until they occupy the positions 1, 2. Used in this way, the stereographic provides in two dimensions a perspective view of the process of rotation which could be carried out on a suitable globe. A similar idea has been applied

[3] Immler, W. Grundlagen der Flugzeugnavigation. Berlin. 1937.
[4] Penfield, S. L. "The Stereographic Projection and its Possibilities from a Graphical Standpoint." *Am. Jour. Sci.* 11 (1901).

to the orthographic projection, in connection with the small scale solution of navigational problems.

The above method provides a " side elevation," as it were, of the process of rotation from the viewpoint of the equatorial axis, while that about to be

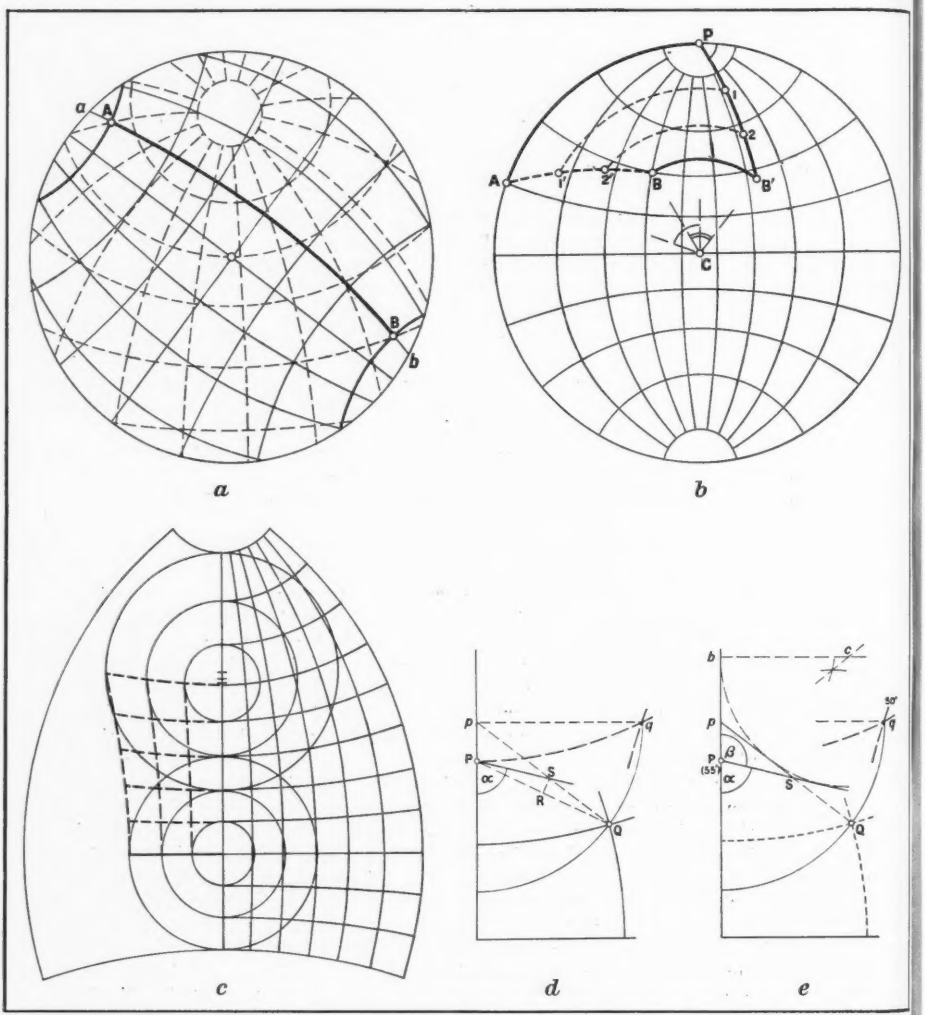

FIG. 2.

Stereographic projection.
Simple methods of adaptation.

described deals with a direct transformation of co-ordinates. If two positions are plotted on an equatorial stereographic projection, it is possible to construct an oblique system of co-ordinates centred on one of the points as pole. The distance and azimuth of the other point are then obtained in terms of oblique

" latitude " and " longitude." Since the pole may be situated at any latitude, it follows that the centre of projection may be given any desired value, and all great circles passing through it will become straight lines.

The basis of the method is illustrated in Fig. 2*c*. A circle centred at any latitude on the sphere will remain a circle on the stereographic projection, but its centre will be displaced along the meridian. If a series of concentric circles be imagined to be drawn on a globe, their centres on the stereographic will be progressively displaced, moving towards the nearer pole as the radius increases. Moreover, a given radius on the globe will vary in length on the projection according to the latitude of its centre. It is, therefore, possible to superimpose on an equatorial graticule a series of curves (shown dotted in the figure) the rectangular co-ordinates of the intersections of which give both the radii of circles and the amount of displacement of their centres, along an initial meridian. The latitude co-ordinate of any point Q on the normal projection may now be expressed in terms of parallels centred on an oblique pole P. This process is illustrated in Fig. 2*d*. The point p must be chosen to fulfil the condition that the perpendicular pq will equal in length the radius pQ. The oblique latitude of Q may then be read off the appropriate curve of equal radius. Since the meridian passing through P and Q will also be projected as an arc of a circle, the longitude may be obtained by the following construction. Join PQ and pQ. Bisect PQ at R, and erect a perpendicular RS, intersecting pQ in S. Join PS. The angle α included between PS and the central meridian will then give the oblique difference of longitude between the central meridian and that of the point Q.

The converse process may also be discussed. If an oblique latitude and longitude and the latitude of the oblique pole are known, co-ordinates of the position on the normal graticule may be determined, although the process is tedious and of comparatively little value. Using the same notation, suppose the co-ordinates 40°N., 60°E. be given in respect of a pole situated at 55°N. Referring to Fig. 2*e*, P will be the pole and p the centre for a radius 50° (co-latitude of 40°). The problem is, to find the position of the intersection S from a knowledge of one side, one angle and the perimeter of the triangle PpS. Mark off Pb equal to half the perimeter (*i.e.*, equal to half Pp plus half the radius of the 40° parallel) and erect a perpendicular bc. Draw Pc bisecting the angle β. Now with centre c and radius cb draw an arc of a circle. If a tangent to this arc be drawn from p it will intersect the parallel of 40°N. in the required position Q, the co-ordinates of which may be read off the normal graticule.

No account has so far been taken of the centre of projection. It is obvious that the position of the pole of any oblique system may be so chosen as to give to this centre any required value. This fact may also be utilised in studying the relations between two points. Thus, let the centre be given the latitude value of a point A. The pole P will then be placed farther north by the amount of the oblique co-latitude of A, and the parallel for the latitude of another point B may now be drawn from P in the usual manner. The position of B on this parallel may be determined either by the construction described above, or from a knowledge of the distance AB or the azimuth of B from A.

A stereographic hemisphere, centred on the equator, can thus be made the basis for a graphical treatment of practically every kind of spherical relation. The central point can be given any required latitude value, and straight lines passing through it will be great circles preserving true bearings. The intersections along either of the main axes provide a scale for the measurement of distances along such great circles. Positions of the oblique pole can be assigned, and, if a scale of centres and radii of small circles (such as is shown in Figs. 2*c* and *d*) is prepared, the parallels of the oblique system can be drawn in as already described. The meridian passing through any point and its pole can be found by rotating these two points (in their same relative positions) until they lie along

a meridian of the hemisphere, when the curve can be drawn in. Convergency is also shown, and is given by the angle PpQ in Fig. 2*d*. Reverse azimuths and lines of equal reverse azimuth can be found with equal facility, as will shortly be described.

Spherical relations can thus be studied with greater convenience, and certainly with greater accuracy, than is afforded even by a globe, and the geometry of navigational problems is well shown. The construction of oblique small-scale stereographic projections also becomes a simple matter,[5] and the normal values of the oblique intersections could be used in the graphical transformation of various other projections for geographical purposes. Fawcett[6] has used the Mollweide very effectively in this way, although his transformations were carried out very tediously on a globe.

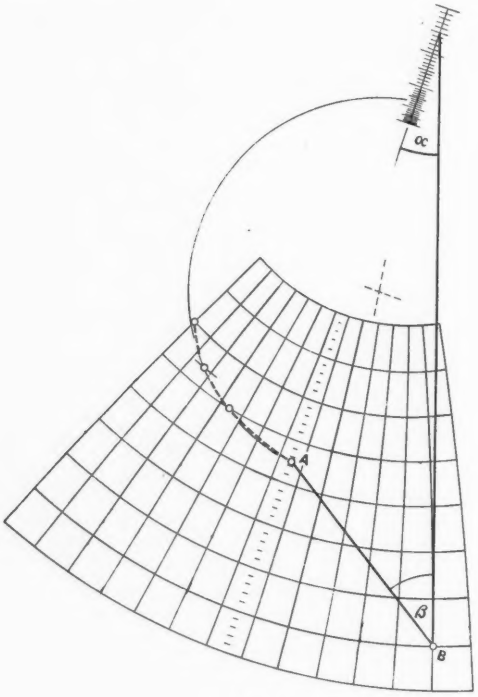

FIG. 3.

For the demonstration of the spherical relations of navigational problems, and for transformation purposes, it would therefore be worth while to construct a suitable device whereby all these operations are reduced to a simple form. The writer has designed an instrument of this kind, in which the various scales described above are combined with a part of a stereographic equatorial hemisphere as base. An upper transparent sheet is pivoted about the centre of this hemisphere, and is graduated around the edge in degrees. Positions plotted on this

[5] This transformation recalls a construction described many years ago (Phillips, A. W., *Rep. Brit. Assoc.*, 1884). This can be made the basis of a kind of pantograph which will trace out an oblique stereographic graticule at one end while a pointer at the other moves over an equatorial one. It would be of little practical value.

[6] Fawcett, C. B. Political Geography of the British Empire. (Univ. Lond. Press.)

sheet can be rotated as desired, and the method of use will be obvious from the foregoing descriptions.

Various properties of the oblique stereographic are brought out in Fig. 3. The great circle distance AB can be measured from A along the central meridian, and the bearing of B from A is true. The reverse bearing is included with a tangent to the meridian of B, which also passes through the centre of the parallel of B. Convergency between A and B is represented by the angle α which this tangent makes with the central meridian. Points on curves of equal reverse azimuth radiating from A are easily laid down by means of the following construction. Having chosen a parallel of latitude through which the curve is to pass, find its centre and describe an arc. Then, taking the initial meridian between the centre of this parallel and the centre of projection (A) as chord, describe an arc in which these points subtend an angle equal to the given reverse azimuth. Arc and parallel will intersect in the required position (Fig. 3).

Suppose now that a series of such oblique projections, each covering the same area, and computed for centres spaced at equal intervals of latitude, be superimposed in a definite order. The various positions of any particular intersection will trace out a curved line, which may be considered to be a locus of the point as it moves in response to a movement of the centre of projection along the central meridian. It is possible so to arrange the projections that these loci are suitably spaced and of a convenient length. They may then be graduated in such a way that the position of any graticule unit will be determined for a given centre by joining corresponding points on the appropriate loci. If a point is plotted within such a unit the distance, azimuth and great circle path between it and the particular centre may be measured along a straight line joining the two points. In this way the most valuable property of the zenithal projections—the preservation of true relations of distance and direction along straight lines passing through the centre of the map—is made universally applicable to the solution of all problems.

Two methods of arranging the projections are possible. If they are positioned in such a way that the various centres coincide, the loci will be of an inconvenient length and will become too complicated for easy reference. The author has made use of this method however, by superimposing the projections themselves (drawn in different colours) on a plate which could be moved $2\frac{1}{2}$°N. and S. of the common centre. This movement was relative to the centre of a circular protractor, which carried a rotatable arm, graduated, according to the stereographic principle, in statute and geographical miles. The projections were constructed at 5° intervals of latitude, and the $2\frac{1}{2}$° movement in either direction ensured that any value, within the range of the projections, could be given to the centre. Other points could be plotted on the appropriate graticule by means of a suitable scale, and distances and bearings could thus be measured with ease. It should be noted that the method is an approximation only, but errors of distance due even to extreme displacement of the centre should not greatly exceed one half per cent. if the radius does not exceed a few thousand miles. The elements of certain of the special projections, etc., in the present paper were obtained in this way, and much tedious labour was thereby avoided.

In the second method of arranging the graticules, the centres do not coincide, but are spaced out along the central meridian in such a way as to provide loci of convenient length. Intersections of parallels on the central meridian are obtained from a series of curves, shown on the left (Fig. 4), while the line ab gives the latitude co-ordinates of centres of projection. To determine the position of a point with reference to any particular centre it will generally be desirable to obtain from the loci the four corners of the graticule unit within which it lies. If these points are joined, and the sides of the resulting " rectangle " are subdivided as required (by means of the scale or dividers shown), the lines joining opposite subdivisions will intersect in the required position. As before, the distance in statute and geographical miles is given on an arm, which may be set

to any desired centre. Bearings are read from a protractor, the right angle of which also serves to define the latitude values of the various centres. This arrangement enables co-ordinates of lines of equal reverse azimuth to be read directly. The instrument is theoretically accurate, and may be constructed on

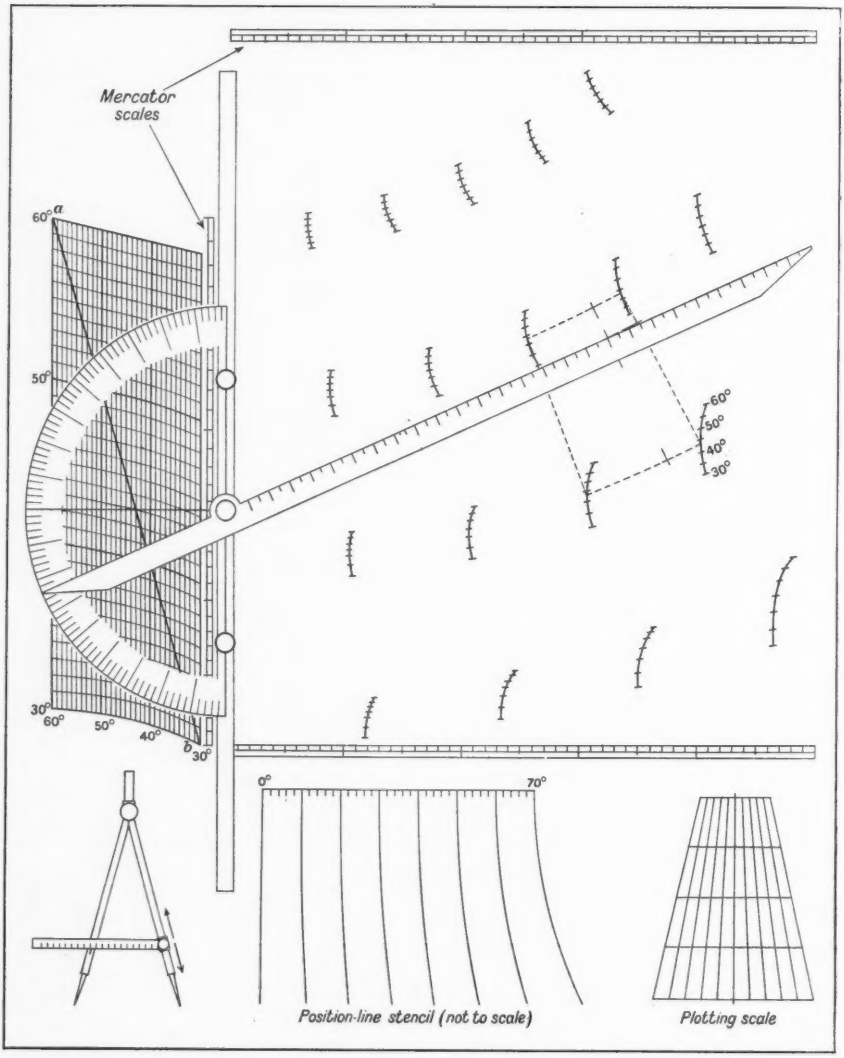

FIG. 4.

Stereographic Great Circle Protractor.

any scale. It would seem to represent the simplest and most logical means of overcoming, by cartographic methods, the fundamental weaknesses inherent in all normal map projections.

GNOMONIC PROJECTION.

Although at first sight this projection would appear to provide a useful medium for adaptation, the loss of conformality is a handicap which outweighs the added advantage of straight line representation for all great circles. In addition to its normal uses for such purposes as the determination of great circle tracks, the plotting of radio bearings by means of distorted protractors, and the representation of long-distance great circle routes by development on regular solids, this projection has been adapted to the solution of the usual problems in distance and bearing.[7]

If on a polar gnomonic projection (Fig. 5) curves are superimposed giving great circle distances along lines perpendicular to an initial meridian, any straight line on the projection may be measured by turning it into this position, when the distance intercepted between the curves may be read off. In the figure the triangle PAB is rotated about P until it occupies a position $PA'B'$. The intercept ab can then be read off to give the distance in any chosen unit of measurement. So far as the measurement of distance is concerned this device is seen to represent the gnomonic equivalent of Penfield's transparent grid of circular arcs, in which the measurement of angle is similarly hampered.

The determination of azimuth is less satisfactory, since it requires both a preliminary measurement of an angle on the map and a conversion to the true value by means of a suitable nomogram. Thus, again referring to Fig. 5, and dealing with the true azimuth (z) of B from A in the spherical triangle PAS, we have (since PA is equal to the co-latitude of A):—

$$\sin \phi = \cot z \cot APS.$$

If the azimuth measured on the projection equals z' the plane angle APS equals $90° - z'$ and the relation:—

$$\tan z = \tan z' \operatorname{cosec} \phi$$

is obtained. From a knowledge of the map azimuth z' the true azimuth (z) can thus be read from a nomogram.

For the purposes of great circle navigation, the Aeronautical Committee of the German Navigation Board has issued four orthodromic charts of this type to cover the following areas:—

Polar Regions.
North Atlantic.
Central Atlantic.
Asia and Europe.

CRAIG'S RETRO-AZIMUTHAL PROJECTION.

First introduced in 1909, Craig's retro-azimuthal principle[8] has since found expression in a class of projections specially adapted to the solution of problems in radio navigation. The azimuthal properties of these projections (like that of Craig), are usually restricted to measurements in respect of a single centre.

It has occurred to the author that the principle could be applied to the determination of true azimuths between *any* two points, by making use of loci of graticule intersections such as have been described above for the stereographic. In Craig's projection, not only the path of the changing centre, but also that of every other point, will fall along its own particular meridian. Correction to longitude is therefore avoided, and the problem reduces to one of determining the amount of movement in latitude which will result from a given change in the position of the centre. The extent of this movement is, of course, controlled by the arrangement of the meridians. Although for the sake of simplicity those of Craig's projection were plotted as equi-distant parallel straight lines, the idea

[7] Immler, *op. cit.*

[8] Craig, J. I. "The Theory of Map Projections." Paper No. 13, Egyptian Survey Dept., 1910.

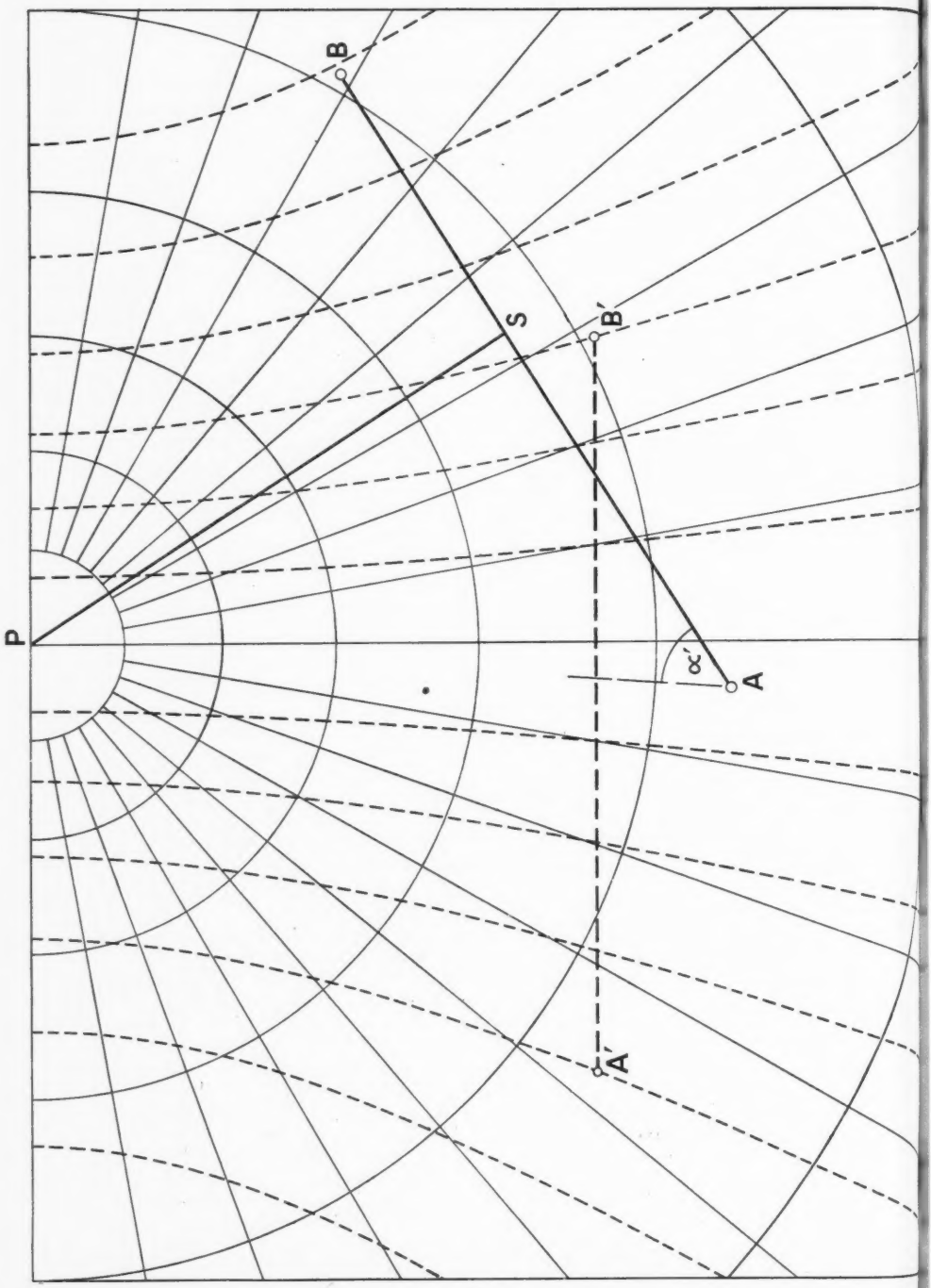

FIG. 5.
Polar Gnomonic Projection, adapted to the measurement of great circles.

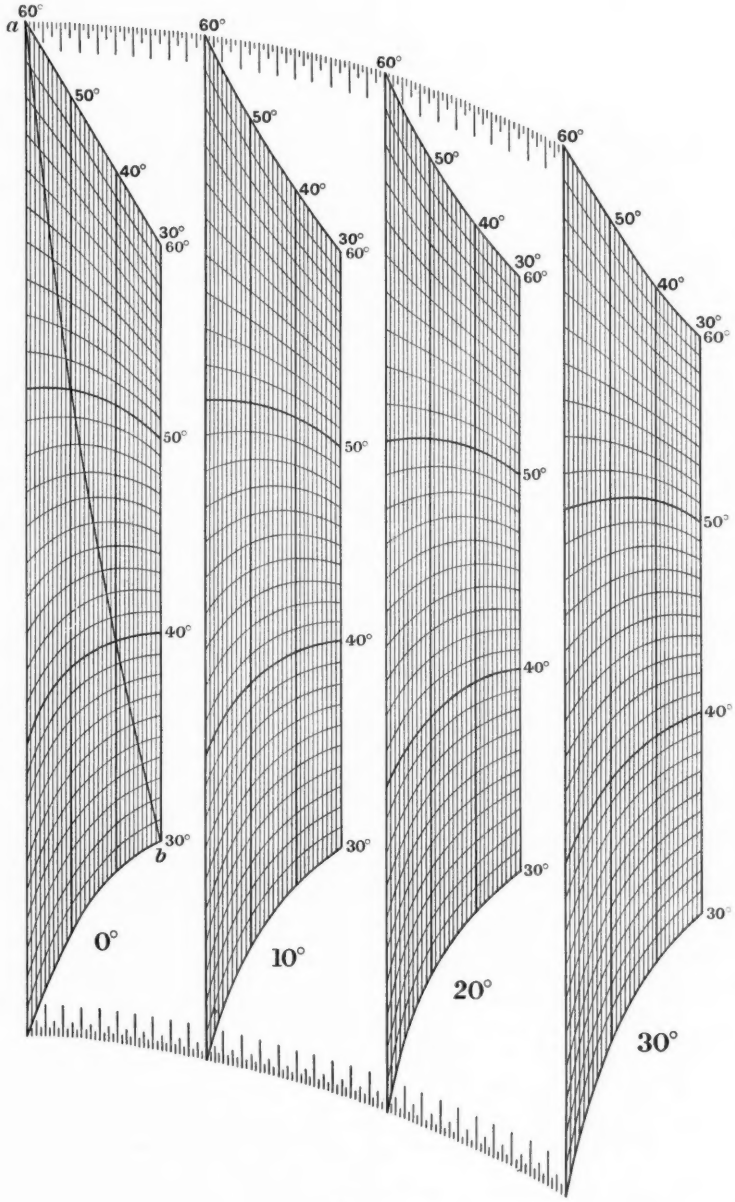

FIG. 6.

Azimuth Diagram based on Craig's Retro-azimuthal Projection.

can obviously be applied (less conveniently, as experiment showed) to meridians of the conical type.

An example of the diagram in its final form, constructed for points situated between latitudes 30° and 60° and allowing for a difference of longitude of 30°, is shown in Fig. 6. It enables true azimuths between such points, as well as the elements of corresponding lines of equal reverse azimuth, to be obtained directly. All meridians remain as equi-distant parallel straight lines, and intersect the parallels in points which fall along four sets of curved loci. The graduation provides for successive changes of 1° in the latitude of the centre, and gives the corresponding intersections for each degree of latitude with every 10° of longitude. Since the scale of longitude remains constant, intermediate meridians can be drawn in by inspection. The actual measurement of the azimuth subtended by a point situated on the initial " meridian " at another point plotted on the projection is best accomplished as follows. The " centre " is plotted at the intersection of its diagonal locus (*ab*) with the required latitude on the " meridian " of 0°. If the " meridians " between which the point is to lie are then chosen, the latitude of the second point may be read from their right hand scales. The corresponding loci are then followed to their intersection with two perpendiculars, the scale readings of which are equal to the latitude of the first point. If a straight line is drawn to join these two intersections (actually the line is slightly curved) the position of the second point upon it is very easily obtained from the uniform scale of longitude.

Two positions have now been determined. If these are joined by a straight line the bearing of the first from the second will be given by the constant angle included with the straight meridians. The co-ordinates of the line itself are of use both in the construction of isoazimuth charts and as an aid to navigation. So far as its purpose is concerned, this type of diagram bears some resemblance to the well-known Littrow projection. The relative advantages of the two methods are more fully discussed below.

Lambert's Conical Orthomorphic Projection.

The projections so far considered depend for their usefulness on the accurate representation of one or more properties of the great circle. Generally speaking, the remaining graticules normally used in map construction introduce variable errors for which allowance cannot easily be made. For this reason adaptation is clearly out of the question, and only where convenience of measurement is combined with a favourable representation of distance and bearing over a wide area can such projections be considered to fall within the present discussion. The conical orthomorphic graticule is the only one which really fulfils these conditions, although that of Albers' Conical Equal-area projection may be associated with it. In both cases straight line distances approximate very closely to the truth, while the use of scale factors gives a further guarantee of accuracy. Even without applying scale factors, errors on the conical orthomorphic should remain within one per cent. over distances of the order of 2,000 miles.

While the path of a great circle does not of course coincide with the straight line joining two points, the departure on this projection in medium latitudes should not greatly exceed one half per cent. of the distance. These statements must be considered in relation to the distribution of lines of strength, which in neither case are regularly disposed about a centre. Generally speaking, the arrangement of lines of equal scale is likely to favour E.-W. axes of measurement, although this is not the case between the standards of Albers' projection. The curves of true scale (isoperimetric curves) which result from the equal-area nature of this projection represent the nearest approach to effective lines of strength, and provide for fairly accurate scaling facilities in approximately N.E.-S.W. and converse directions.

The representation of azimuth is less easy to define. Although the average error is kept within reasonable limits there is considerable variation in different

directions. Some idea of this variation may be gained from Fig. 7, where errors in azimuth from, and reverse azimuth to, a selected point (lat. 30°N.) are shown graphically for both projections. The angles subtended at this point by positions on a normal projection and the corresponding positions shown in pecked lines will in each case equal the error in bearing.

The close approximation given by the conical orthomorphic is evidently sufficient for many navigational purposes, and the projection is clearly of use for medium-scale aeronautical maps in several sheets. In fact, it may be said to be the only potential rival of the Mercator as a base map for navigation.

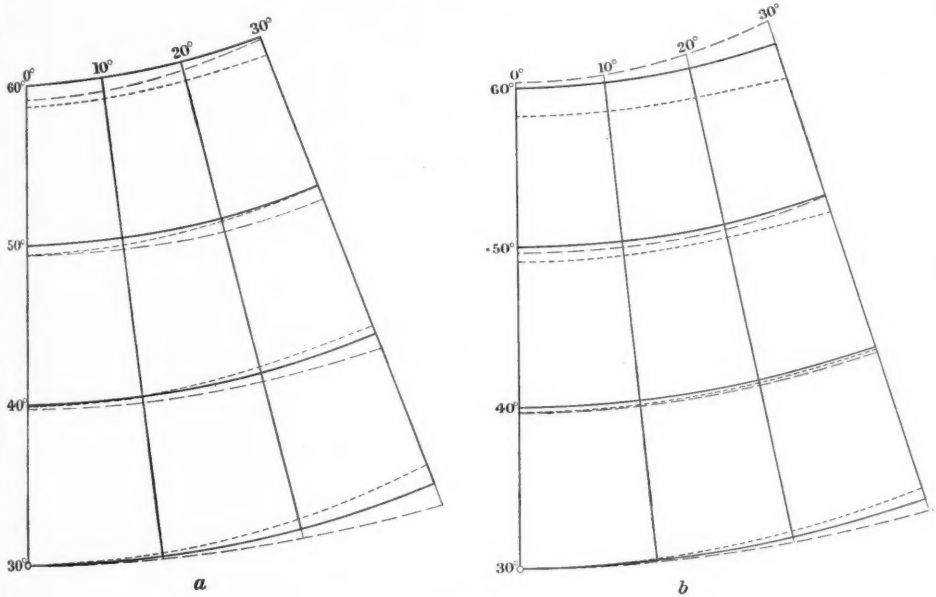

FIG. 7.

——— *Conical Orthomorphic Projection.*
– – – – *Azimuthal Projection (30° centre).*
- - - - *Retro-azimuthal Projection (30° centre).*

——— *Conical Equal-area Projection.*
– – – – *Azimuthal Projection (30° centre).*
- - - - *Retro-azimuthal Projection (30° centre).*

MERCATOR'S PROJECTION.

In spite of the increased range of methods available, the Mercator chart has retained its traditional place as a basis for the cartographic work of the navigator. This is due as much to the fact that the various auxiliary methods of measurement presuppose the existence of a base map for the plotting of navigational data as to the continued use of the loxodrome for this type of work. The relation of the Mercator to the present discussion is therefore to be considered largely from the point of view of supplementary methods developed to assist the normal functions of the projection.

The well-known relation between great circle and loxodromic bearings, utilised on Mercator charts for the plotting of both great circle courses and radio position-lines, introduces as a spherical problem the determination of convergence. Although there are several graphical methods of simplifying the calculation from the formula (Diff. long. sin mid-lat.) the stereographic is the only projection

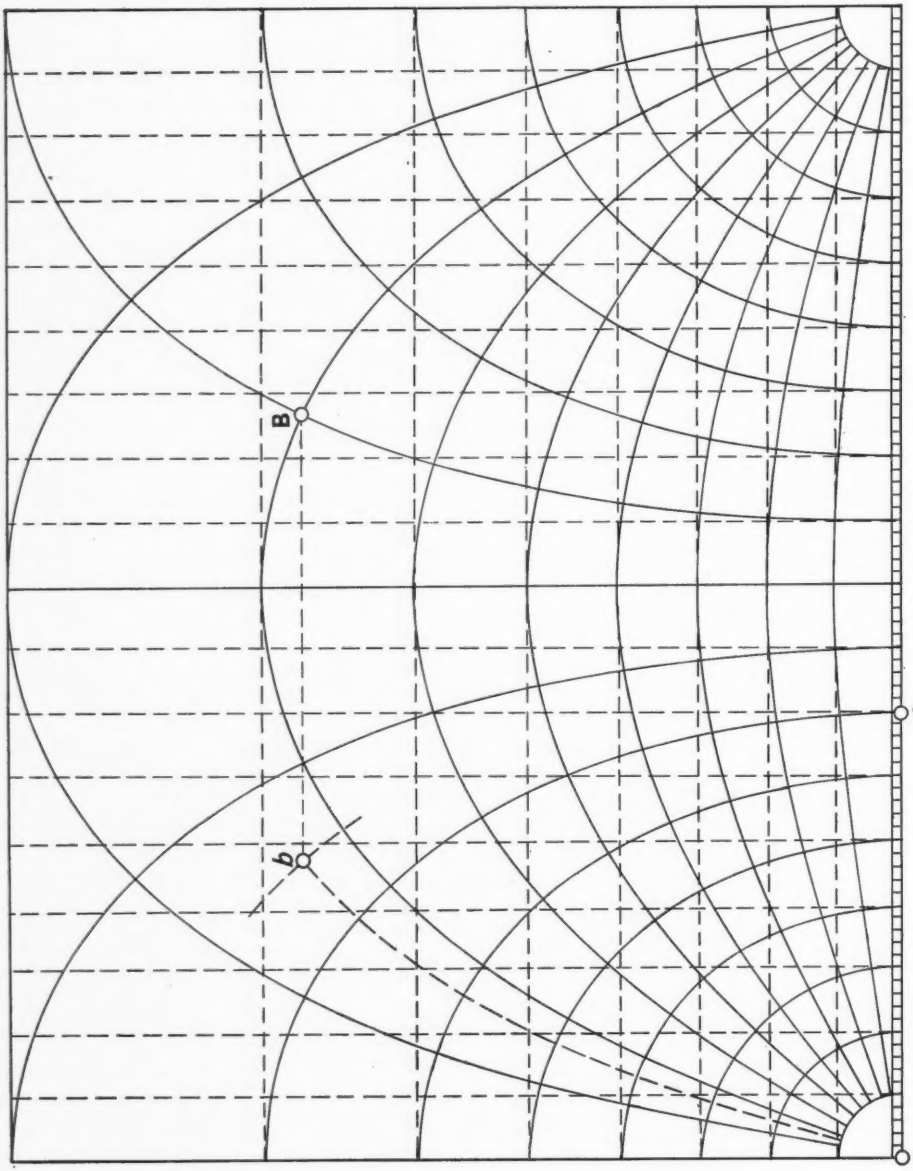

FIG. 8.

Transverse Mercator Projection,
showing use for measurement of distance and bearing.

providing a convenient cartographic solution. The possibilities in this direction, which have already been discussed, depend on the fact that the stereographic is both azimuthal and retro-azimuthal. In practice, values for convergence are most readily obtained from a simple nomogram.

Loxodromic distance and bearing are required for comparison between rhumb line and great circle courses, and are also obtained most conveniently from suitably constructed nomograms.[9] The nomogram is entered with diff. long./diff. lat. on the first scale, and with mid-latitude on the third. The intersection with the middle scale of a line joining these two points will give the required bearing and distance factor. This factor must be multiplied by the mid-latitude figure to give the loxodromic distance.

The application of the stereographic to the rotation of positions about a perpendicular axis has its counterpart in the Veater Diagram used in long distance air navigation to transform declination and hour angle into altitude and azimuth. In this case the Gauss Conformal (Transverse Mercator) Projection is used, and the turning of positions through a given angle, accomplished on the Reeves Azimuth Diagram by actual rotation, is here effected by movement along straight lines parallel to an initial meridian. Although this method is generally confined to the solution of Sumner line problems, there is no reason why it should not be extended to provide small scale determinations of distance and bearing. Fig. 8 illustrates the principle involved. If the distance AB and the azimuth of B from A are required, the point B is moved along a line parallel to the normal equator Aa by an amount equal to the initial co-latitude Aa on the transverse projection. The distance is then given by the co-latitude ab and the azimuth by the longitude of b. A large-scale projection of an octant of the sphere may be drawn and divided for convenience into a number of sheets. In this way considerable accuracy is obtainable. The same idea has been put into mechanical form by Immler,[10] the transverse projection being wrapped around a cylinder which may be turned through any given angle (*i.e.*, the co-latitude) to bring any required altitude and azimuth value opposite a reference mark on an outer box.

FIG. 8.
Transverse Mercator Projection, showing use for measurement of distance and bearing.

Certain possibilities of the oblique Mercator have recently been explored by Mr. A. R. Hinks,[11] who shows how well the land surface of the earth can be depicted if the pole is situated at 28°N., 165°E. As before, the transverse projection is used for the measurement of distance, and also for laying down great circle tracks. If a transverse case is superimposed on an oblique one, with the main axes in coincidence, it is merely necessary to see that the two points on the latter whose relations are required are aligned along a meridian of the former, when the portion of the curve intercepted by the points will be the trace of the great circle joining them, and distance will be given by the difference between the readings of the orthogonal system of curves. Bearings can be obtained by plotting on the transverse case, as shown in Fig. 8. A similar method could of course be adopted in respect of the normal projection. The whole object of using the oblique case is for the opportunity it gives to exhibit " unusual " world relations.

Finally, the ease of construction of the Mercator graticule must not be overlooked. The projection is completely defined if a scale of latitude and an equally divided scale of longitude are given. It is therefore possible to include these scales in certain navigational devices, so that rhumb lines and mercatorial bearings may be projected if required without the aid of an actual graticule. Thus it is only necessary to add a suitable latitude scale to the Azimuth Diagram already mentioned to enable loxodromes as well as isoazimuths and mercatorial as well as true bearings to be measured. The same idea may evidently be applied to the Great Circle Protractor (again based on loci). In this case the arm (set

[9] Immler, *op. cit.*
[10] *op. cit.*
[11] Hinks, A. R. *Geog. Jour.* 97 (1941).

to 90°) would be used to define latitude, while the protractor would measure mercatorial bearings of points so plotted.

* * * * *

Before turning to the second type of problem certain special cases must be given a passing notice. In addition to the ordinary gnomonic, the two-point case of this projection may be of occasional use, as in finding (in one operation) the origin of wireless signals received at two stations. The Two-point Equi-distant projection is an analogous case, in which true distances may be measured along straight lines passing through two selected points. These straight lines do not, however, coincide with the paths of corresponding great circles. In each case the graticules are suitable for map construction. On the other hand, the Retro-azimuthal Equi-distant projection described by Mr. Hinks[12] is entirely unsuitable for this purpose, although measurements of distance and reverse azimuth in respect of a single point can be made under the same conditions as exist on a normal zenithal projection.

b. Solution of Problems Involving One Unfixed Position.

Generally speaking, the cartographic work concerned in the fixing and checking of position during long-distance flights has to deal with the laying down of loci obtained as a result of celestial and terrestrial observations of altitude, azimuth or difference of azimuth. The following geometrical relations define the observations which may usefully be made for this purpose:—

1. The locus of a point A situated at a given distance from another point B is a circle centred on B and with radius equal to AB.
2. If two points A, B, subtend a given angle at a third point C, the locus of C is the arc of a circle constructed on AB as chord and passing through C.
3. If two points A and B appear to coincide when viewed from a third point C the locus of C is the straight line passing through A and B.
4. The locus of a point A which subtends a given bearing at another point B is the line including the given bearing with the meridian of B.
5. The locus of a point A at which a point B subtends a given bearing is the curve joining points at which lines passing through B make angles equal to the given bearing with the local meridian.

The practical expression of locus No. 1 is, of course, the well-known Sumner line, deduced from an astronomical observation of altitude and azimuth. On the basis of No. 2 it is possible to obtain a line of position by an observation of the difference of bearing subtended at a dead reckoning position by two known points, such as two radio stations. The plotting of such lines may be simplified with the help of suitable projections. With regard to No. 3, it is possible to make use of an observation of the transit of a star across the vertical of another star, the " vertical " in this case being a line perpendicular to the visible horizon. The position of an observer at the time of observation would be somewhere along a great circle passing through the two substellar points.

Another form of position line is obtained by measuring the bearing of a heavenly body or of a terrestrial point such as a radio station. In this case the locus will take the form either of a great circle or of an isoazimuth, according as the bearing is measured at the known position of the radio station or at the undefined position of an observer. This is illustrated in Fig. 9. If the bearing (α) of a ship or aeroplane B is measured at a radio station A the line of position for B is obviously the great circle AB (No. 4 above). But if the bearing β of station A is measured at B (an unknown position) the position line for B will be

[12] Hinks, A. R. " A Retro-azimuthal Projection of the whole Sphere." *Geog. Jour.* **73** (1929.)

the isoazimuth AB (shown as a broken line) which is the locus of points where great circles through A make angles equal to β with the local meridian (No. 5 above). To a fair approximation and for distances up to 1,000 miles, both lines may be considered to have equal and opposite curvature, so that the isoazimuth is a great circle that has been " turned over."

In order that these relations may be put to practical use, the cartographic expression of the various loci must as far as possible be reduced to a regular geometrical form. The study of this type of spherical problem is therefore concerned on the one hand with the construction of graticules on which great circles, isoazimuths and lines of equal difference of azimuth are given either by straight lines or by circular arcs, and on the other with the laying down of sections of arcs of equal zenith distance obtained by observation of heavenly bodies.

Dealing first with terrestrial observations, the methods at present in use may be briefly mentioned, in order that certain new matter which follows may be related to standard practice. So far as radio bearings are concerned, the Littrow

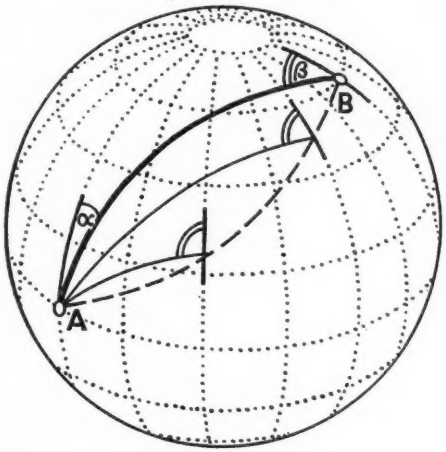

FIG. 9.

Projection bears much the same relation to lines of equal reverse azimuth as the gnomonic does to the great circle. Radio stations are plotted on a central meridian, and reciprocals of all reverse azimuths are laid off from these stations to give the straight paths of corresponding isoazimuths. In addition to its use in transferring from great circle to rhumb line bearings, and vice-versa, the relation between loxodrome and great circle also enables position lines determined by directional wireless to be laid down with some approach to accuracy on projections of the Mercator or conical types. Thus, if a position line is of the great circle variety, it is only necessary to apply the approximate conversion angle to the initial bearing and to plot the resulting loxodrome on a Mercator chart. At the intersection of this line with the dead reckoning longitude (or latitude) a tangent to the great circle may be drawn at an angle equal to half the convergence. This tangent will approximate to the required line of position. In the case where the bearing is measured at a dead reckoning position, an isoazimuth may be approximately determined in the same way. If a Mercator chart is used the conversion angle must be applied to the reciprocal of the bearing. This time a tangent to the *isoazimuth* will be drawn through the intersection of the loxodrome with the dead reckoning longitude, making an angle equal to

half the convergence with the former. The same method may be employed on projections such as the conical orthomorphic. As in this case the straight line represents an approximate great circle, the angle it includes with the tangent will equal the convergence, and not the conversion angle as before.

Certain new methods and projections, designed to facilitate the solution of problems of this kind, are dealt with below. They include in the Azimuth Diagram based on Craig's Projection and the Two-Point Retro-azimuthal Projection, two ways of projecting isoazimuths as straight lines. In addition, Maurer has devised a graticule in which these loci become arcs of circles passing through the nearer pole. Another convenient method (not previously described) enables the co-ordinates of lines of equal reverse azimuth to be determined through the medium of the Stereographic Great Circle Protractor. It has been shown above that any stereographic graticule which includes a scale of centres for each parallel can be used for the same purpose. Lines of equal difference of azimuth also possess certain advantages when used in conjunction with other methods. Projections on which they become arcs of circles are quite easily constructed for any two chosen points.

Sumner lines, or circles of equal altitude, are less amenable to cartographic treatment on account of the long distances involved. If an oblique system of geographical co-ordinates be described about the geographical position of a heavenly body as pole, it may be considered to move obliquely over the surface of the earth in response to changes in declination and hour angle. The sight-working of the navigator is directed to laying down on a chart a section of one of the " parallels " of such a system. The representation on a map of a locus of this kind can take the form either of a circle, as on the stereographic (see below) or of an irregular curve. Although its representation on the Mercator takes the latter form, advantage has been taken of the rectangular conformal nature of this projection to reproduce in special devices one component of the movement over the earth of small circles of constant altitude. These curves are projected on to the chart, and allowance can be made for changes in position due to the earth's rotation. In this way a locus based on any observation of altitude and azimuth is determined in the neighbourhood of a D.R. position by purely graphical means.

Any alternative method of placing an astronomical line of position on the navigation map has first to overcome the difficulty that the standard of accuracy required prohibits the appearance on the same sheet of the D.R. position and the subsolar or substellar point. If it were not for this fact the problem would resolve itself into a comparison between the measured distance and bearing and an observed altitude and azimuth. The alternative is to compute values for the D.R. position or a point situated near it, and to step off from this reference point along the azimuth the difference between the computed and observed zenith distances. A perpendicular drawn through the point so obtained will be a tangent to the circle of position.

Graphical methods similar in principle to the Veater Diagram already mentioned provide the only examples of a use of projections in the computation of altitude and azimuth. Non-cartographic methods of solving this problem include complicated instruments like the Spherotrigonometer and the Nautical Calculating Machines of Immler-Askania, as well as many similar devices. The Veater Diagram (really a part of a Transverse Mercator projection) enables both the D.R. position and the geographical position (declination and hour angle) of a heavenly body to be plotted and turned through the co-latitude of the former to give the altitude (distance) and azimuth between the two. The advantage of Mercator's projection for this purpose depends on the rectangular nature of the graticule, which may therefore be divided into a number of large-scale sheets to give results comparable with those to be obtained from a mathematical solution of the spherical triangle.

MAURER'S ORTHOMORPHIC RETRO-AZIMUTHAL PROJECTION.

CRAIG'S RETRO-AZIMUTHAL PROJECTION (SPECIAL ADAPTATION).

STEREOGRAPHIC GREAT CIRCLE PROTRACTOR.

These methods are alike in that they are designed to deal with lines of equal reverse azimuth radiating from any point on a central meridian. As was stated above, the Littrow Projection is the most widely used device of this kind. It suffers from a certain disadvantage, however, in that the inclination of the meridians (which are slightly curved) is the reverse of that in most normal projections. Maurer's graticule, which belongs to the conformal class, succeeds in overcoming the disadvantages of this divergence in high latitudes. It is so constructed that isoazimuths radiating from any point on an initial meridian become arcs of circles passing through the nearer pole. To draw such a locus

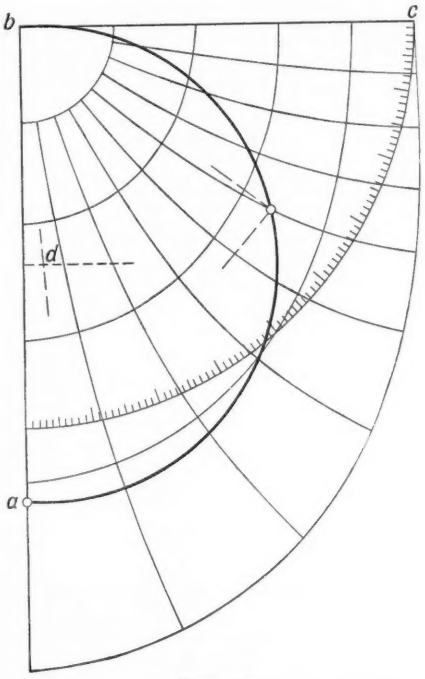

FIG. 10.

Maurer's Orthomorphic Retro-azimuthal Projection, showing method of plotting isoazimuths.

for a station *a* (Fig. 10) it is only necessary to bisect the co-latitude *ab*, and to lay off from *bc* the given reverse azimuth (0°-180° E. or W.) on the protractor. The intersection at *d* of bisector and angular radius (which really functions as a meridian) gives the centre of the required circle (*ba*). As in the case of the Littrow Projection, a second arc can be drawn if desired and displaced by its co-ordinates through the appropriate difference of longitude. The point of intersection with the first arc will fix the observer's position.

A suggested adaptation of Craig's Retro-azimuthal Projection has already been described. Its use in plotting isoazimuths follows from the construction. Straight lines passing through the centre intersect every meridian at the same

angle, which is the true reverse azimuth from any point on the line. Such lines are therefore isoazimuths, and the purpose of this adaptation is to make these properties applicable to lines drawn through *any* latitude on an initial meridian. The determination of the co-ordinates of other positions is a simple matter once the user has become accustomed to the basic idea. It should be noted that this device is not subject to the limitations of the Littrow Projection in high latitudes, although the distance between parallels increases rapidly north of latitude 60°.

The Stereographic Great Circle Protractor may also be employed to determine points on lines of equal reverse azimuth. Imagine the central " meridian " to represent successively meridians at intervals of $X°$ from the sending station. A locus of the position of the sending station relative to each such intermediate meridian must be plotted before each determination is made. If this has been done for the first point, the arm must be set to indicate the given reverse bearing on the protractor. It is now required to find by trial a latitude on the central meridian at which this arm will pass through the same latitude on the locus of the sending station. This is affected by moving the protractor until a coincidence is found. The latitude so determined, together with an even longitude already chosen, constitute the co-ordinates of a first point on the required line of position.

Two-Point Retro-Azimuthal Projections.

If the properties of the preceding projections are restricted to lines passing through two points (in general not on the same meridian) a " fix " may be obtained at once. The two-point gnomonic, in which bearings included between straight lines (great circles) passing through two selected points are correctly preserved, has already been mentioned. It is evident that the same idea may be applied to reverse azimuths. These may be measured on the projection either at the stations which subtend them or at points at which they are subtended. The first of these methods (Fig. 11*a*) is the more successful, since straight lines laid off from the stations at reciprocals of the reverse bearings will intersect in the position of the observer. In this way a " cut " is obtained at once, without the necessity of determining loci. As will be seen later, a position line based on difference of azimuth can also be determined with equal ease.

Provided a table of azimuths, or a suitable instrument for measuring them is available, the graphical construction of a two-point projection of this kind is quite a simple matter. Two points, A and B, are plotted on a straight line to represent two radio stations. A straight meridian is drawn through the point A making with AB an angle equal to the reverse azimuth of A measured at B. Similarly, the angle included with the meridian of B equals the reverse azimuth measured at A. Intersections of meridians and parallels are fixed by laying off at each station the reverse azimuth subtended at the point required. Straight meridians and curved parallels (convex to the nearer pole) are then drawn through the points so found to complete the graticule. These intersections are subject to a lack of symmetry which becomes more marked with increase in the longitude range of the projection, and which would disappear completely if the " centres " were situated either on the same parallel or on the same meridian. For a longitude range in medium latitudes of anything up to 30°, and for a difference of latitude between the " centres " of not more than a few degrees, this discrepancy is not excessive, and may be averaged out without introducing any serious error. At the same time, the two points selected should as nearly as possible be situated on the same parallel. If circular protractors are added, the position of the observer is found at once by the intersection of two rays. A trial projection of the N. Atlantic area, constructed by the writer in this way, was found to provide a satisfactory graticule.

As would be expected, the alternative method of laying down the reverse bearings results in a projection (Fig. 11*b*) not greatly different in appearance from the conical orthomorphic. The angle included between the meridians of two " centres " again equals the convergency between these two points. In

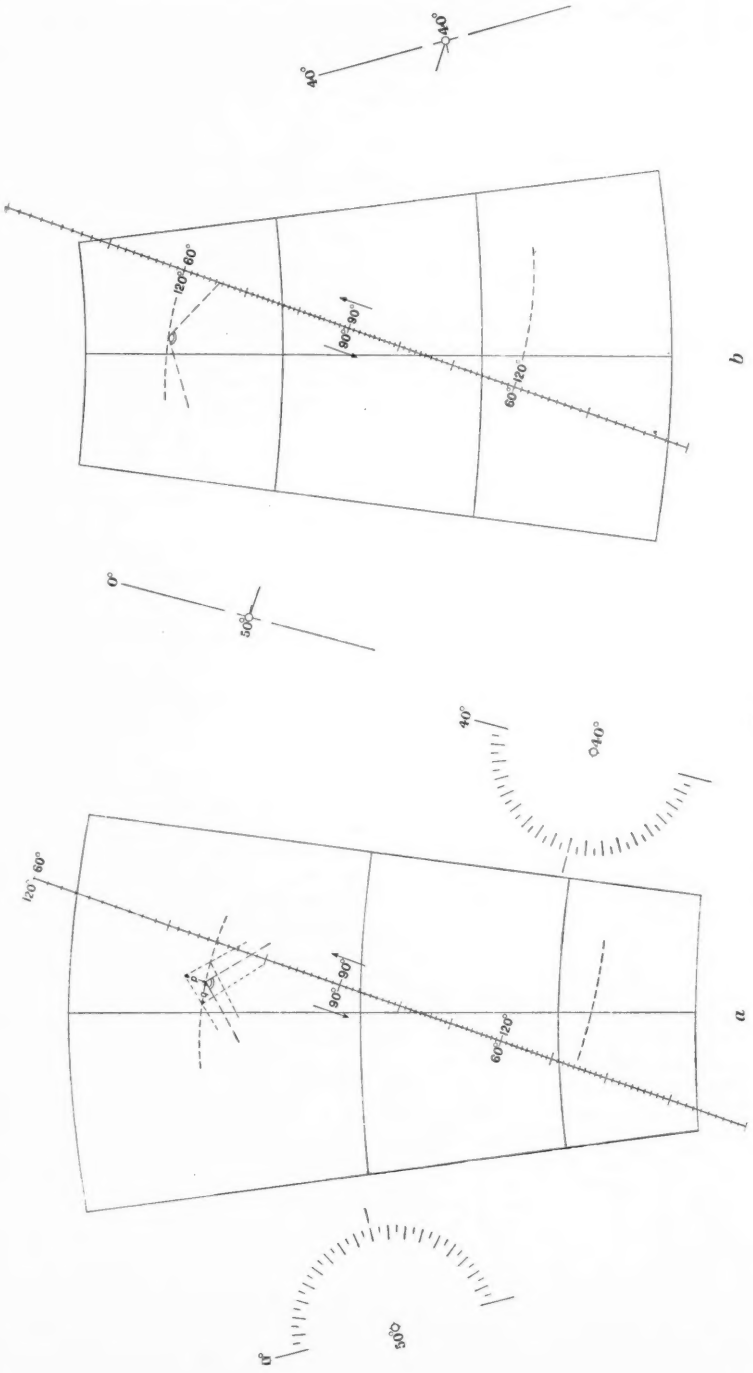

FIG. 11.

Two-point Retro-azimuthal Projections.

this case, however, the framework of the preceding projection is, as it were, turned upside down, while reverse azimuths are laid off from the positions at which they are measured. A lack of symmetry is again noticeable, and the projection suffers from the same limitations as the previous one, without offering any advantages in use.

These two-point graticules recall the use of curves of equal reverse azimuth on Mercator's projection. In this case co-ordinates of suitable loci are determined for two stations and plotted as intersecting systems of curves. Positions are thus fixed by inspection, and the method of use is essentially similar to that of the two graticules described above. It should, however, be noted that, in the first of these latter, points at which reverse azimuths of a " centre " are equal fall on the same straight line.

Projections Preserving True Difference of Azimuth.

If, instead of two reverse azimuths, their difference is given, a line of position may clearly be obtained. On both the two-point retro-azimuthal projections this locus will take the form of an arc of a circle. In the one case, where bearings are measured at points at which they are subtended (Fig. 11*b*), the normal relations of plane geometry will apply, and the centres of circular arcs may be located by inspection on a suitable scale. This scale (shown in the figure) is very easily constructed along the perpendicular bisector of a straight line joining the two radio stations. If from one of these points (taking the direction of the other station as zero) rays are drawn at intervals of one degree, their intersections with the bisector furnish the graduations of the scale. Each division is numbered on one side as the complement of the angle laid off, and on the other as the sum of the angle itself plus 90°. This graduation gives the centres of circles in which the two stations subtend given differences of bearing, angles included in the two segments being supplementary.

A similar line of position can be determined on the converse type of projection. In this case also the scale is constructed along the perpendicular bisector of a line joining two " centres." Although corresponding arcs will pass through similar positions on the graticule, angles subtended within these arcs will not in general agree with the values indicated on the scale. The actual construction of this scale, however, remains quite a simple matter. If a ray is laid off from each of the two " centres " the position line corresponding to the sum of these bearings (*i.e.*, to the total difference of bearing) will be a circle described about the stations and passing through the intersection of the rays. To complete such a scale it is only necessary to find by trial the centres of a number of these circles. It will be found that distances between graduations will agree with those of the previous scale, although the values of these divisions will not agree. In the example given (Fig. 11*a*) the whole scale is in effect displaced some distance to the south.

So far as radio bearings are concerned this type of locus is clearly independent of any constant error, which may arise from a variety of causes. A " fix " can be obtained either from the intersection of two arcs (making use of three or four stations) or from one arc and an isoazimuth drawn for one of the bearings. Even where two isoazimuths are employed it might be worth while to determine the azimuth-difference position line as a check.

In cases where a constant error of bearing ($x°$) occurs, the displacement along an arc may be calculated from a knowledge of x and of the distance between the stations. If the errors, though still equal to x, are of opposite sign (*i.e.*, it they have the effect of increasing or decreasing the total difference of bearing) the displacement will be approximately perpendicular to that along the arc. Letting p and q denote the respective amounts of displacement along these axes due to errors of bearing, the possible positions of an observer who knows nothing of the sign of either error will be somewhere within the quadrilateral

defined by the diagonals $2p$ and $2q$. Immler[13] has pointed out that the possible error in position will be smallest when the difference of azimuth equals 90°, *i.e.*, when the quadrilateral becomes a square and is therefore of minimum area. The two-point projection of such a quadrilateral of position is shown in Fig. 11*a*, where it is drawn for a possible error of 2° in either ray.

STEREOGRAPHIC PROJECTION.

Since all circles are preserved in their true form this projection has an obvious use in dealing with Sumner lines. The geometry of circles centred on an initial meridian has already been dealt with in the first part of the present paper. A small-scale device there described would appear to be of value in demonstrating the Sumner line principle, although the scale of construction is necessarily too small for actual use. For practical purposes a large-scale graticule, specially computed for a chosen reference point, has been suggested. This reference point will be arbitrarily chosen at an even latitude, and may be employed for all D.R. positions in its neighbourhood. Computed azimuths may then be laid off in straight lines, and the difference between computed and observed zenith distances, measured in terms of the radial scale, will be stepped off along it. Immler's suggestions for the use of this method include a number of devices from which values of altitude and azimuth may be obtained for the reference point in question. These are based on special nomograms, and include the Immler Altitude Protractor, a nautical calculating machine (Immler-Askania) and a micro-calculator (Immler-Askania).

Advantage has been taken of the properties of the stereographic to prepare a special template (Immler Altitude Stencil), which gives the curvature of arcs of equal altitude for 10° intervals and to a given scale. These curves are used to draw in position lines, which will be perpendicular to the azimuth and always concave to the substellar point. The Stereographic Great Circle Protractor may evidently be fitted with a template of this kind. Reference points can then be chosen at any latitude, and difference of altitude will be measured along a scale of geographical miles given on the arm, which will also indicate true azimuths.

3. SUMMARY.

It remains to give some indication of the relative value of certain of the methods and instruments already described. These may be arranged in two main groups, the basis of division being the scale to which they are best adapted. Thus, a small-scale group, including methods based on a rotation or transformation of co-ordinates, may clearly be distinguished. In most cases the scale of construction is of necessity too small for navigational work, and devices of this kind will be of more use for geographical and demonstration purposes. They depend entirely on a use of co-ordinates and cannot conveniently be used in conjunction with map detail. In the large scale group determinations will be restricted to a range of a few thousand miles. Here it is necessary to distinguish between normal projections combining convenience of measurement with representation of ordinary detail, and those adapted solely for the purpose of linear or angular measurement. The one type includes graticules like the stereographic (used in the form suggested by Penfield) as well as those of the two suitable conical projections. Penfield's device is the most useful for long distances, while for a range of less than 1,000 miles Immler's method of using the same projection is to be recommended. The second type obviously includes the Great Circle Protractor and the Azimuth Diagram described above.

This use of scale as a basis of division is obviously bound up with the question of standards of accuracy required for different purposes. In other words, the accuracy relevant to problems concerned with really long distances is generally such as to allow small scales to be used. This relation, however, does not hold

[13] *op. cit.*

in the case of the navigator's use of Sumner lines. Here cartographic methods of solution are greatly limited by the impossibility of making precise measurements over the distances involved. Under such conditions the rivalry between cartographic and non-cartographic methods is intensified, and elaborate mechanical devices for the solution of the spherical triangle come more into their own. In the majority of cases, however, geographical (*i.e.*, small-scale) problems are well within the range of cartographic solution. Errors of the order of anything up to one per cent. of the distance may often be tolerated in such cases.

The solution of large-scale problems is a different matter. Here the range of distance is restricted to between a few hundred and a few thousand miles and the accuracy desired is correspondingly greater. Under such conditions, the two conical projections often provide surprisingly good results so far as the measurement of distance is concerned. The measurement of bearings on a normal conical projection remains an intractable problem if errors of 1° at, say, 2,000 miles are considered excessive. This type of work evidently depends on principles already well established, such as the correct use of scale factors, the choice of standard parallels and the kind of allowance to be made for the spheroid. Where greater convenience and precision are required, special adaptations supply a much-needed want, and allow almost every kind of measurement to be made under conditions of theoretical accuracy.

Since the need for large-scale adaptations is mainly due to the cartographic problems of the navigator, the errors implicit in aerial dead reckoning may be taken to be sufficiently typical of other problems of the same kind. Thus, in a distance of 500 miles an error in position of, say, 15-20 miles is not exceptional. When it is remembered that, in the air, horizontal angles (whether in the form of compass readings or radio bearings) are not to be relied on within a degree or so, and vertical angles within a few minutes of accuracy, these figures are seen in their true perspective. Coupled with the extreme mobility of aircraft and the absence of material obstacles these limits combine to render a fairly high degree of approximation both necessary and permissible under such conditions. So far as astronomical observations are concerned, the method of working and plotting for position should provide for a final result that is accurate to the nearest tenth of a degree. For preliminary determinations, errors of distance should not greatly exceed one half per cent. and bearings should be computed to the nearest half degree. A knowledge of the difference between great circle and loxodromic distances allows a choice to be made between a course flown on a single bearing and one composed of " chords " of the great circle. It will usually be found that the saving due to the latter procedure will not be worth while unless the difference between the two initial courses is considerably more than 5°, as the value of the percentage increase in distance will be roughly equal to the value of such small angular differences.

The actual plotting entailed during flight is usually carried out on a base map (generally on Mercator's Projection) while the various accessory quantities and conversions are obtained from suitable auxiliary charts (gnomonic, etc.). It remains to see to what extent various special devices already described may usefully replace these normal methods. Immler's recommendations for this purpose[14] include an elaborate course recorder based on a circular stereographic chart covering some 10° in latitude. These charts are interchangeable and may be turned about a centre to indicate any desired course. A pointer, moved by a suitable mechanism, travels along the course line at a speed equal (on the scale of the map) to that of the 'plane. Distances along such stereodrome courses may also be indicated. By the addition of a suitable curved stencil the instrument may be converted into an altitude position line plotter. Although this ingenious device may be of considerable use it would appear to err on the side of over-

[14] *op. cit.*

elaboration, since little more than a mechanical time scale is provided, and various auxiliary methods will still be required.

While nearly all the methods dealt with in the present paper are capable of use as adjuncts to the navigator's base chart, only one (the Great Circle Protractor based on loci) combines the various functions in a single instrument of convenient form. In addition to providing simple and theoretically accurate solutions of all problems in distance and bearing on any required scale of construction (necessary in view of any possible change of plan after the flight has begun) the instrument may also be used in the same way to give the co-ordinates of points on lines of equal reverse azimuth. It has been suggested above that Mercator elements could be made available by the addition of simple scales of latitude and longitude, while it might also be possible to include similar scales for the measurement of loxodromic distance. As this instrument is also based on the stereographic it is equally capable of being used for the plotting of Sumner lines. Reference points for this purpose may be situated at any latitude. In this way all the spherical elements required for navigational purposes are obtainable from a single device, which may be constructed for any area within the limits required, and to any scale.

As recently as 1934, Deetz and Adams[15] were able to write: "Possibilities of the extensive use of the gyroscopic compass and the employment of radio signals in the fixing of positions by wireless directional bearings present new problems in the navigation of ships and aircraft. The various uses and tests to which nautical and aerial charts are subjected may still further try the patience of the mathematical cartographer in devising more suitable systems of projection, or in supplying short-cut mathematical or automatic expedients as convenient adjuncts to the Mercator chart." The present paper is an attempt both to survey this field and to contribute to the efforts which have been made to supply the need foreshadowed in these words. At the same time, it should not be forgotten that this navigational viewpoint represents only a single aspect of a subject which must assume even greater importance when the effects of modern travel technology have inspired or compelled some attempt at world planning and a world outlook.

ACKNOWLEDGMENT.

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[15] Deetz, C. H., and Adams, O. S. "Elements of Map Projection, with Applications to Map and Chart Construction." Special Pub. No. 68, U.S. Coast and Geodetic Survey.

REVIEW.

AERONAUTIKA TERMINARO.

By E. D. Durrant. Internacia Esperanto-Ligo (Rickmansworth, Herts). 1941. Price 10d., post free.

Any review of a book in Esperanto tends to degenerate into a controversy regarding the merits of that " ersatz " language. The reviewer holds strong views on the subject, but feels that this is not the place to express them. Admitting the necessity of a universal second language, the possibility of imposing it, and the suitability of Esperanto for the purpose, there can be nothing but praise for the way in which Mr. Durrant has compiled his glossary of aeronautical terms. Weighing just under 1 oz., and small enough to slip into any picket, this booklet gives over 400 technical expressions, with definitions in Esperanto (based on the British Standards Glossary) and English equivalents. The classification is practical, and the printing has been done with commendable accuracy.